SAT-based Cryptanalysis of Cryptographic Hash Functions

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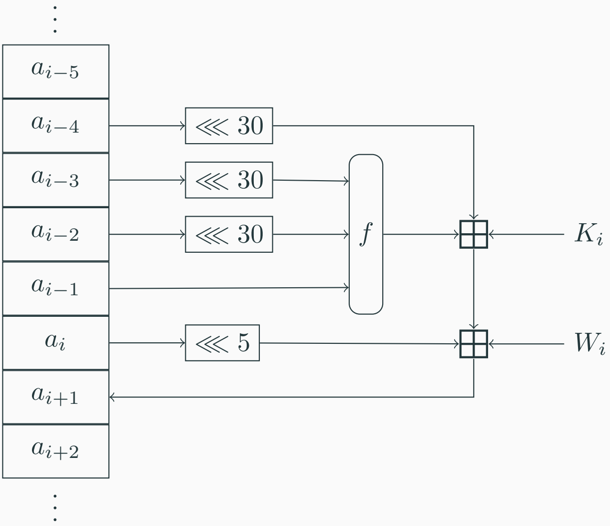
Problem statement

- **Algebraic Fault Attack on SHA-1 function**: Find an input message given the hash output and extra constraints extracted from hardware implementation of the hash
- Core problem:
  - **Preimage attack**: Given H, find x such that $Hash(x) = H$.
  - Encode SHA-1 function in CNF, fix the output variables to H

SHA-1's Step Function



a_i, K_i, W_i: 32-bit vectors; f: 32-bit bitwise function

SHA-1 Bitwise function

$$f(x,y,z) = \begin{cases} IF(x,y,z) = (x \wedge y) \vee (\neg x \wedge z), & 0 \leq r \leq 19 \\ XOR(x,y,z) = x \oplus y \oplus z, & 20 \leq r \leq 39 \\ MAJ(x,y,z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z), & 40 \leq r \leq 59 \\ XOR(x,y,z) = x \oplus y \oplus z, & 60 \leq r \leq 79 \end{cases}$$

Encoding Basic Blocks in CNF

$$f = IF(x, y, z) \equiv \bigwedge_{i=0}^{31} f_i \leftrightarrow (x_i \wedge y_i) \vee (\neg x_i \wedge z_i)$$

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\begin{aligned}
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&\equiv \bigwedge_{i=0}^{31} (f_i \vee \neg x_i \neg y_i) \wedge (f_i \vee x_i \vee \neg z_i) \wedge \\
&\qquad (\neg f_i \vee x_i \vee z_i) \wedge (\neg f_i \vee \neg x_i \vee y_i)
\end{aligned}
$$

Encoding Basic Blocks in CNF

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\begin{aligned}
f = XOR(x,y,z) &\equiv \bigwedge_{i=0}^{31} f_i \leftrightarrow (x_i \oplus y_i \oplus z_i) \\
&\equiv \bigwedge_{i=0}^{31} (\neg f_i \vee \neg x_i \vee \neg y_i \vee z_i) \wedge (\neg f_i \vee \neg x_i \vee y_i \vee \neg z_i) \wedge \\
&\qquad (\neg f_i \vee x_i \vee \neg y_i \vee \neg z_i) \wedge (\neg f_i \vee x_i \vee y_i \vee z_i) \wedge \\
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f = MAJ(x,y,z) &\equiv \bigwedge_{i=0}^{31} f_i \leftrightarrow (x_i \wedge y_i) \vee (x_i \wedge z_i) \vee (y_i \wedge z_i) \\
&\equiv \bigwedge_{i=0}^{31} (\neg f_i \vee x_i \vee y_i) \wedge (\neg f_i \vee x_i \vee z_i) \wedge (\neg f_i \vee y_i \vee z_i) \wedge \\
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Multi-operand modular addition

$$a_{i+1} = (a_i \lll 5) \boxplus W_i \boxplus K_i \boxplus f(a_{i-1}, a_{i-2}, a_{i-3}) \boxplus a_{i-4}$$

[Nos12]: Nossum, SAT-based Preimage Attacks on SHA-1

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\hline
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 & z_3 & z_2 & z_1 & z_0 \\
 & t_3 & t_2 & t_1 & t_0 \\
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  - No unit clause to propagate!

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[GOS$^+$12]: Ganesh et al., Lynx: A Programmatic SAT Solver for the RNA-Folding Problem, SAT12

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- Encoding the problem in CNF+C

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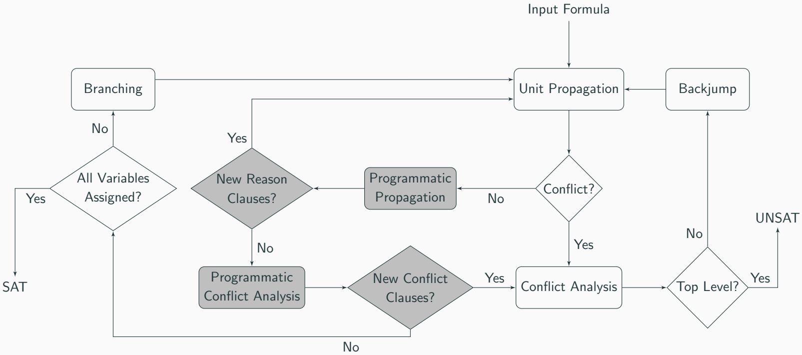
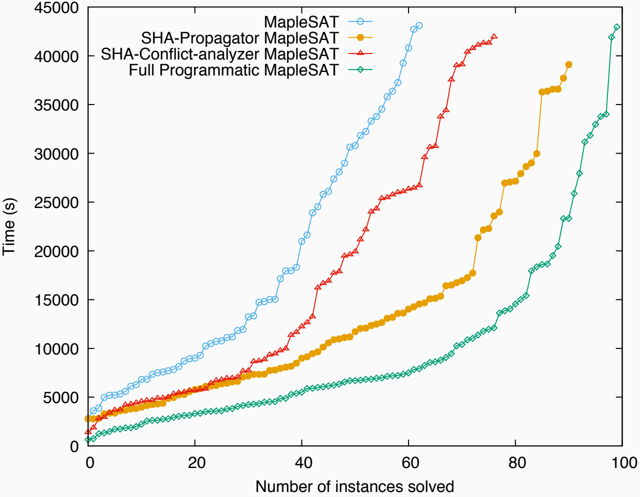
Programmatic SAT



I/O Arc-consistent

- We added reason clauses for input/output relations
- Unit+Programmatic propagation: input/output arc-consistent
- Also called "propagation complete" in some papers

Results



[NHGG18]: Nejati et al., Algebraic fault attack on SHA hash functions using programmatic SAT solvers, CP18

Thanks!

Questions?

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